On the construction of Chern–Simons terms in the presence of flux

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Abstract

We develop a method for relating the boundary effective action associated with an orbifold of the D+1–dimensional theory of a p–form field to D–dimensional fluxed Chern–Simons type of terms. We apply the construction to derive from twelve dimensions the Chern–Simons terms of the eleven dimensional supergravity theory in the presence of flux.

1 Introduction

Compactifications of the superstring theories to lower dimensions in the presence of non-trivial background R–R and NS–NS fluxes are the subject of active current research interest. Motivated first from being a possible solution to the hierarchy problem [1], they revealed a rich theoretical and phenomenological structure in compactified $D = 4$ low energy effective field theories. In particular, flux compactifications offer the possibility of lifting all moduli in lower dimensions and possess semi–realistic cosmology in the presence of a positive cosmological constant [2, 3]. The main effort in flux compactifications has been concentrated earlier on various compactifications of M–theory [4], more recently on compactifications of type IIB superstring theory on six dimensional (generalized) Calabi–Yau manifolds and toroidal orientifolds (for a review see [5]) but much less is known about type IIA superstring flux compactifications. Recently, the compactification of type IIA supergravity to four dimensions where the compact space is a Calabi–Yau manifold or a T^6/Z_3 orientifold was studied in detail [6, 7]. There it was shown that in these models all geometric moduli can be stabilized classically. The latter results give the strong hint that flux compactifications of the type IIA theory can be at least as interesting as the ones of type IIB.

In this context, it seems particularly interesting to address the issue of the construction of the Chern–Simons terms of type IIA supergravity in the presence of background flux. Let us recall that in the absence of flux the Chern–Simons terms $H_3 \wedge C_3 \wedge F_4$ and $-B_2 \wedge F_4 \wedge F_4$ (we use the notation of [8]) are equivalent, since the total derivative vanishes on the boundary. However, this is not the case in the presence of flux and therefore the definition of Chern–Simons terms becomes more subtle [6, 7]. This can be seen by noticing that Chern-Simons terms in the presence of (topological) flux are, in general, not invariant under (large) gauge transformations. The correct definition of a Chern–Simons term Γ on a D dimensional manifold M_D in the presence of topological flux is

$$\int_{M_D} \Gamma = \int_{M_{D+1}} d\,\Gamma, \tag{1.1}$$

where the D–dimensional manifold M_D is a boundary of a $D+1$ dimensional manifold M_{D+1}. However this is not enough in order to resolve the ambiguity. Only if the (by construction) gauge invariant right hand side can be reduced onto the boundary in a smooth, gauge invariant way, then one can obtain the most general Γ, consistent with D–dimensional gauge invariance. In this way one can construct a general class of Chern–Simons terms (i.e. excluding the terms containing curvature forms which are present in the type IIA theory [9, 10]) relevant for flux compactifications.

2 The orbifold construction

Here we will present a method of obtaining the type of Chern–Simons terms discussed above by employing the $U(N)$ orbifold gauge theory construction of [11]. The idea is to encode the non–trivial topology and any symmetry breaking occurring at the orbifold fixed point into the transition functions associated with the gauge fields living on those charts that contain the orbifold fixed point in their intersection. Then, shrinking the intersection to a point (the fixed point), if the value of the transition function at that point is non–zero, the boundary effective theory will develop contributions that look like flux effects along the boundary. Even though for concreteness we will discuss only the case of 11–dimensional supergravity, we believe that the method can be in general applied to the problem of construction of gauge invariant, fluxed Chern–Simons terms in any dimension.

Let us therefore consider 11–dimensional supergravity and try to find its most general flux extension. To do so, we will follow [12] and [7] and introduce flux in the 11–dimensional action via a 12–dimensional manifold with boundary. We start by assuming the existence of a 12–dimensional manifold $\mathcal{M}_{12}$ on which a 3–form field can be locally defined. The coordinates on this manifold are $z^M = (x^\mu, x^{11})$. The 11–dimensional coordinates are $x^\mu = (x^{\hat{\mu}}, x^{10})$. We would like then to construct the Z_2 orbifold of this theory by projecting out by the reflection $\mathcal{R}$, which acts on the 12-dimensional coordinates as

$$\mathcal{R} z = \overline{z}, \qquad \overline{z} = (x^\mu, -x^{11}). \tag{2.2}$$

The coordinate x^{11} can be either space–like or time–like. The action of $\mathcal{R}$ on a rank–r tensor field $C(z)$ is defined as

$$\left(\mathcal{R}\, C_{M_1 M_2 \cdots M_r}\right)(z) = \alpha_{M_1} \alpha_{M_2} \cdots \alpha_{M_r}\, C_{M_1 M_2 \cdots M_r}(\mathcal{R}\, z)\,, \tag{2.3}$$

where no sum on the M_i is implied on the right hand side. The intrinsic parities are defined by $\alpha_\mu = 1$ and $\alpha_{11} = -1$. Parity of the exterior derivative of forms can be easily derived using that $[\mathcal{R}\,, \partial_M] = 0$. At the fixed point of the orbifold, $x^{11} = 0$, an 11-dimensional theory living on the boundary manifold $\mathcal{M}_{11}$ can be consistently defined. We would like this theory to be somehow related to the 11–dimensional supergravity of [13].

We compactify x^{11} on a circle of radius R. [1] The gauge invariant construction of the orbifold proceeds by defining separate 3–form $U(1)$ gauge fields on overlapping charts that provide an open cover for the 12–dimensional space. The minimum number of such overlapping open sets in the $x^{11} = 0$ neighborhood is two, let us call them $O^{(+)}$ and $O^{(-)}$

[1] By taking the radius of the circle to infinity one can describe in this way the non-compact version of the orbifold.

and their overlap $O^{(+-)} = O^{(+)} \cap O^{(-)}$. On each open set there is a 3–form gauge field that under a 12–dimensional $U(1)$ gauge transformation transforms with its own 2–form gauge function

$$\text{on } O^{(\pm)} : \qquad \delta\, C^{(\pm)} = d\,\Lambda^{(\pm)}. \tag{2.4}$$

One requires that the 3–forms on $O^{(+-)}$ (where they are both defined) are related by a gauge transformation:

$$C^{(+)} = C^{(-)} + d\, g^{(+-)}, \qquad C^{(-)} = C^{(+)} + d\, g^{(-+)}\,. \tag{2.5}$$

The $U(1)$–valued 2–forms $g^{(+-)}$ and $g^{(-+)}$ are transition functions and they are defined on the overlap of charts $O^{(+-)}$. The gauge invariant 4–form field strength

$$G = d\, C^{(+)} = d\, C^{(-)}, \qquad \delta\, G = 0, \tag{2.6}$$

does not depend on the chart label since $d^2 = 0$ and therefore it is uniquely defined throughout $\mathcal{M}_{12}$. The Bianchi identity $d\, G = 0$ asserts that G is closed and so defines an element of $H^4(\mathcal{M}_{12}, R)$, the fourth cohomology class of $\mathcal{M}_{12}$ with real coefficients.

The consistency of the system of equations Eq. (2.5) requires the existence of a 1-form χ defined on $O^{(+-)}$ such that

$$g^{(+-)} + g^{(-+)} = d\, \chi. \tag{2.7}$$

Furthermore, the form of Eq. (2.5) is preserved under the gauge transformations

$$\begin{aligned} \delta\, g^{(+-)} &= \Lambda^{(+)} - \Lambda^{(-)} + \frac{1}{2}\, d\, (\lambda^{(+-)} + \lambda^{(-+)}) \\ \delta\, g^{(-+)} &= \Lambda^{(-)} - \Lambda^{(+)} + \frac{1}{2}\, d\, (\lambda^{(+-)} - \lambda^{(-+)}), \end{aligned} \tag{2.8}$$

for some 1–forms $\lambda^{(+-)}$ and $\lambda^{(-+)}$. Combining Eq. (2.7) with Eq. (2.8) we derive the gauge transformation of χ

$$\delta\, \chi = \lambda^{(+-)} + d\, \phi_0 \tag{2.9}$$

for some 0-form ϕ_0. The above is essentially an appropriate generalization of a 1-form fibre bundle construction to higher order forms, known as gerbes, see for example [14].

In order to introduce flux in the boundary theory, one must construct a non-dynamical 4–form on the overlap $O^{(+-)}$, let us call it $\overline{G}$, so that in the limit where the overlap shrinks to a single point (the orbifold fixed point) it goes to an x^μ-dependent function, which is in general not zero (examples of analogous constructions in Yang–Mills theories have been constructed in [11]). If in addition this 4-form obeys

$$d\, \overline{G} = 0, \qquad \delta\, \overline{G} = 0, \tag{2.10}$$

then it can be safely added to the gauge invariant field strength G. Its effect at the orbifold fixed point will be a non-zero flux on $\mathcal{M}_{11}$. One such 4-form is

$$\overline{G} = \overline{g} \wedge \overline{g}, \qquad \overline{g} = g^{(+-)} + g^{(-+)} \tag{2.11}$$

provided that $\lambda^{(+-)}$ is exact, i.e. $\lambda^{(+-)} = d\, l_0$, that is provided that $\overline{g}$ is gauge invariant. Since $\overline{G} = d\,\overline{C}$, with $\overline{C} = \chi \wedge d\chi$, for a gauge invariant $\overline{g}$ under a gauge transformation we have $\delta\overline{C} = d((l_0 + \phi_0) \wedge d\chi)$.

Next, we have to define the action of the reflection operator on the geometry and the fields. For concreteness take

$$O^{(+)} \;=\; (-\epsilon, \pi R + \epsilon) \quad \text{and} \quad O^{(-)} \;=\; (-\pi R - \epsilon, \epsilon) \tag{2.12}$$

with overlap $O^{(+-)} = (-\epsilon, \epsilon)$, where $0 < \epsilon < +\pi R/2$ (we concentrate on the neighborhood of $x^{11} = 0$ only). The reflection operator maps $\mathcal{R}\, O^{(\pm)} \;=\; O^{(\mp)}$, $\mathcal{R}\, O^{(+-)} \;=\; O^{(+-)}$. The transformation $\mathcal{R}$ can be defined also to act on tensor fields defined on $O^{(\pm)}$ giving as result tensor fields defined on $O^{(\mp)}$. On the overlap, we define

$$\mathcal{R}\, C^{(+)} = C^{(-)}. \tag{2.13}$$

We are interested in the action of $\mathcal{R}$ on the fields $\overline{g}, \chi, \lambda^{(+-)}$ and ϕ_0. It is not hard to check that one can consistently define

$$\begin{aligned} &\mathcal{R}\,\overline{g} = \overline{g}, && \mathcal{R}\,\chi = \chi \\ &\mathcal{R}\,\lambda^{(+-)} = \lambda^{(+-)}, && \mathcal{R}\,\phi_0 = \phi_0. \end{aligned} \tag{2.14}$$

Furthermore, if we choose $\overline{g}$ to be gauge invariant then

$$\mathcal{R}\, l_0 = l_0. \tag{2.15}$$

As far as the projections that the above actions imply, it is only the components of the 3-form field and its field strength along the boundary that survive.

The boundary theory is now simple to obtain. One is instructed to construct all possible gauge invariant terms using the original 12-dimensional fields, G and $\overline{G}$ in our case and then take the limit where the overlap $O^{(+-)}$ shrinks to the fixed point of the orbifold action. The limit $\epsilon \longrightarrow 0$ can be taken with the only essential ingredient needed being that

$$\lim_{\epsilon \longrightarrow 0} \overline{G} \equiv G^{flux}(x^\mu) \neq 0 \tag{2.16}$$

and a similar condition on the limit of $\overline{C}$ (in a way that quantum effects do not trigger the appearance of new dynamical fields on the boundary). The result is simple for the terms involving G. The 12–dimensional interaction term is

$$S_{12}^{int} = -\frac{1}{2\kappa_{12}^2} \int_{\mathcal{M}_{12}} (G + \alpha\overline{G}) \wedge (G + \beta\overline{G}) \wedge (G + \gamma\overline{G}), \tag{2.17}$$

where α, β and γ are, to this end, arbitrary coefficients in the absence of a symmetry that can relate them. On the boundary, it reduces to the Chern–Simons interactions

$$S_{11}^{CS} = - \frac{1}{12\kappa_{11}^2}\int_{\mathcal{M}_{11}} \Big\{\hat{C}\wedge\hat{G}\wedge\hat{G} + \alpha'\,\hat{C}\wedge\hat{G}\wedge G^{flux} + \beta'\,\hat{C}\wedge G^{flux}\wedge G^{flux} + \gamma'\,C^{flux}\wedge G^{flux}\wedge G^{flux}\Big\}, \tag{2.18}$$

where $\hat{C}$ and $\hat{G}$ are essentially the 12–dimensional 3–form and 4–form evaluated at $x^{11} = 0$.

Let us emphasize that the procedure described above does not determine the coefficients α', β' and γ'. In order to obtain their precise values, either a direct computation of boundary counterterms or the implementation of some extra symmetry is needed. Regarding the first option, given that in a field theory boundary counterterms appear beyond the classical level [15] and their computation requires a good control of the theory at least at the perturbative level, it seems out of reach at present for any 12–dimensional theory. This difficulty suggests one to consider the possibility of supersymmetrizing the theory since in supersymmetric theories it is not uncommon that coefficients that appear in the effective action at the quantum level are completely fixed by means of symmetry. In the case of eleven dimensional supergravity these coefficients can be fixed from knowing that the full Lagrangian which, besides Eq. (2.18), includes also an Einstein–Hilbert action and a kinetic term for the three form, must be a bosonic part of an effective action describing the eleven dimensional supergravity in the presence of fluxes. The quantity C^{flux} can be now interpreted as a solution of the bosonic equations of motion of the eleven dimensional supergravity while $\hat{C}$ is a quantum fluctuation around it. In this interpretation the flux part of the effective action in general can depend on the coordinates of M_{11} and can break a part of supersymmetries. Requiring agreement with the Chern–Simons term obtained in [7] fixes

$$\gamma' = 0, \tag{2.19}$$

as well as

$$\alpha' = \beta' = 3. \tag{2.20}$$

The choice of γ' is natural since it means that one omits a term in the effective action which corresponds to a non–zero vacuum energy. The coefficients α' and β' (the value of G^{flux}) can be alternatively fixed by compactifying the theory on $S1 \times CY_3$ and matching the resulting four dimensional theory to a specific $\mathcal{N} = 2$ gauged supergravity theory [6, 7].

It would be nice though if it was possible to fix such coefficients directly via the higher dimensional theory imposing e.g. appropriate supersymmetry transformations. Unfortunately, it is difficult to say how the explicit realization of supersymmetry in dimensions

higher than eleven (see [16] for the review on higher dimensional superalgebras) works. It is also known that in dimensions higher than eleven one necessarily obtains fields of spin larger than two, for which no analogue of the Einstein–Hilbert action is known. However let us note that the eleven dimensional superalgebra

$$\{Q_\alpha Q_\beta\} = P^\mu \gamma_{\alpha\beta} + \gamma^{\mu\nu}_{\alpha\beta} Z_{\mu\nu} + \gamma^{\mu_1\cdots\mu_5}_{\alpha\beta} Z_{\mu_1\cdots\mu_5} \tag{2.21}$$

can be embedded in the 12–dimensional $\mathcal{N}=1$ superalgebra with signature $(10,2)$ in a Z_2 parity invariant way. Alternatively it can also be obtained by orbifolding the 12–dimensional $\mathcal{N}=2$ superalgebra

$$\{Q^i_\alpha Q^j_\beta\} = (\tau_a)^{ij}(\gamma^{MN}_{\alpha\beta} Z^a_{MN} + \gamma^{M_1\cdots M_6}_{\alpha\beta} Z^{a+}_{M_1\cdots M_6}) + \epsilon^{ij}(C_{\alpha\beta} Z + \gamma^{M_1\cdots M_4}_{\alpha\beta} Z_{M_1\cdots M_4}), \tag{2.22}$$

where $i=1,2$ and $a=1,2,3$. Note also that the supermultiplet which is a representation of these twelve dimensional algebras does contain a three form field [17], which is expected to reduce to the eleven dimensional three form after the orbifold projection along the lines we have described. It is therefore not unlikely that the whole action obtained by supersymmetrizing Eq. (2.17) reduces to some generalization of the 11D-supergravity action upon orbifolding. Finally, a possible mathematical handle on the nature of the 12–dimensional multiplet could be the fact that it is the lowest lying Euler multiplet [18] associated with the symmetric space $E_6/(SO(10)\times SO(2))$ [19].

3 Conclusion

We presented a possible method for constructing gauge invariant flux extensions of Chern–Simons terms in D–dimensions via an orbifold construction. This can be achieved by formulating the theory on a manifold of dimension $D+1$ of which the original manifold is a boundary, as suggested in [12]. Gauge invariant $D+1$–dimensional fields and gauge transformation functions can be smoothly pulled back onto the boundary, defining a theory automatically gauge invariant in the D–dimensional sense. Remnants of a certain class of bulk gauge transformation (transition) functions are seen as flux along the boundary.

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